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AN OPTIMAL DESIGN FOR THE MAXIMUM FUNDAMENTAL FREQUENCY OF LAMINATED SHALLOW SHELLS

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Abstract—An optimal design is presented to determine the stacking condition that maximizes the lowest frequency of a laminated composite shallow shell with rectangular planform. The shallow shell considered has a symmetric laminate and is supported by shear diaphragms along the four edges. An analytical solution for natural frequencies is derived by discarding the cross-elasticity terms and then solving the governing equations of motion. In the optimization, fiber orientation angles in the layers are introduced as design variables, and a set of differential equations are presented which satisfy the Kuhn–Tucker optimality conditions. Formulas are then derived to give the possible optimal fiber orientation angles. Using numerical examples, the effects of various shell curvatures upon the optimal fiber orientation angles are discussed for a wide range of shallow shell configurations. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The use of fibrous composite materials has advanced extensively not only in the aerospace industry but also in automobile, civil and other fields of engineering. The long fiber composites are typically used in the form of laminated plates and shells, which are known to allow designers to control or tailor the mechanical properties, for instance, by considering fiber orientation angles and stacking layer thickness. In that aspect, effective optimization techniques are of great importance in the design process of laminated composite plate and shell structures. General optimization techniques were overviewed in some references, e.g., in a review paper written by Olhoff and Taylor (1983). Various examples were presented by Vanderplaats and Weisshaar (1989) to demonstrate the use of optimization to tailor composites to meet specific requirements. Eleven papers were collected, with special emphasis on optimization of composite materials, in a book edited by Marshall and Demuts (1990).

The literature survey shows that there are a reasonable amount of technical papers dealing with optimization of laminated flat plates. For vibration problems, Bert (1977) considered design of a composite plate which maximizes the fundamental frequency, and Reiss and Ramachandran (1987) presented a paper to derive a closed form optimal solution for the rectangular plate. Recently in the 1990's, Narita and Ohta (1993) proposed a simplified optimal design method by discretizing design variables for the problem, and Fukunaga *et al.* (1994) used a lamination parameter approach to solve for optimal fiber orientation angles. It is needless to say that more papers are found for cases where other object functions are focused, such as stiffness optimization (Fukunaga and Vanderplaats, 1991) and buckling optimization (Haftka and Walsh, 1992).

Very little has been done, however, to study optimization of shell components or structures, even when the topic is not limited to vibration and/or laminated composites. Plaut and Johnson (1986) used the geometric curvature as a design variable to determine the shell form to maximize the frequency of the shells. Levy and Spillers (1989) considered a cylindrical shell problem of finding the wall thickness that maximizes the buckling load. A recent paper written by Mota Soares *et al.* (1995) seems most related to the topic of the present paper. They presented a two-level design approach to deal with composite plate/shell

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structures by sequentially using the Davidon–Fletcher–Powell (DFP) and the modified feasible direction methods for optimization, and using the finite element method for structural analysis part. The reason why past relevant literature is so limited on the optimization for vibration of composite shells is quite obvious, since the vibration analysis of composite shells involves itself a formidable task in the formulation and a joining optimization algorithm naturally includes many parameters and complicated algorithm flows.

The main contribution of this work is to formulate an analytically oriented design method to optimize the free vibration behaviours of laminated shallow shells. For a structural analysis part, an exact frequency solution is derived for symmetrically laminated shallow shells supported by shear diaphragms along the four edges. An optimization part is formulated by stating that the fundamental frequency parameter is an object function to be maximized and the fiber orientation angles are design variables. A set of equations are then derived, through the sensitivity analysis, to yield possible solutions for the optimal fiber orientation angles by making a direct use of Kuhn–Tucker optimality (stationary) condition. Through a number of numerical experiments, two distinct cases of how optimal solutions exist were classified when the fundamental mode shape is not self-evident. In examples for shallow shells with three types of different curvature, the present design method is demonstrated to have a wide applicability without numerical instability. The effects of the shell curvatures on optimal solutions are clarified, which are not observable in the optimization problems of flat plates.

2. VIBRATION SOLUTION OF LAMINATED SHALLOW SHELLS

Figure 1 shows a shallow shell of rectangular planform with the side length $a \times b$. The shell has a quadratic middle surface with relatively small, constant principle curvatures $1/R_x$ and $1/R_y$, where R_x and R_y are the curvature radius in the x and y directions, respectively. The thickness h is constant and the twisting curvature is not considered here $(1/R_{xy} = 0)$. A symmetrical laminate is considered, thereby neglecting the coupling stiffness $(B_{ij} = 0)$. The cross-elasticity terms are also ignored $(A_{16} = A_{26} = D_{16} = D_{26} = 0)$ by focusing on a relatively large number of angle-ply layers.

Under these assumptions, the following governing equations are reduced from a complete set of general equations for shallow shells (Leissa and Qatu, 1991).

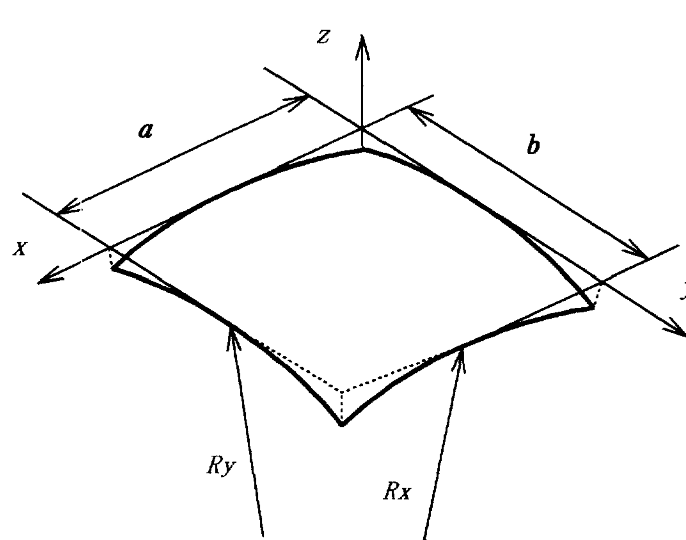

Fig. 1. Shallow shell geometry (neutral plane) and coordinate system.

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \frac{\partial^2}{\partial t^2} \begin{bmatrix} -\rho & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \tag{1}$$

where $u(x, y, t)$ and $v(x, y, t)$ are inplane displacements of a point on the middle surface in the x and y directions, respectively, and $w(x, y, t)$ is an out-of-plane displacement in the z direction. The ρ is a mean mass per unit area of the shell. The elements L_{ij} in eqn (1) are differential operators given by

$$L_{11} = A_{11} \frac{\partial^2}{\partial x^2} + A_{66} \frac{\partial^2}{\partial y^2}$$

$$L_{12} = (A_{12} + A_{66}) \frac{\partial^2}{\partial x \, \partial y}$$

$$L_{13} = \left(\frac{A_{11}}{R_x} + \frac{A_{12}}{R_y} \right) \frac{\partial}{\partial x}$$

$$L_{22} = A_{66} \frac{\partial^2}{\partial x^2} + A_{22} \frac{\partial^2}{\partial y^2}$$

$$L_{23} = \left(\frac{A_{12}}{R_x} + \frac{A_{22}}{R_y} \right) \frac{\partial}{\partial y}$$

$$L_{33} = D_{11} \frac{\partial^4}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x^2 \, \partial y^2} + D_{22} \frac{\partial^4}{\partial y^4} + \left(\frac{A_{11}}{R_x^2} + 2 \frac{A_{12}}{R_x R_y} + \frac{A_{22}}{R_y^2} \right) \tag{2}$$

In eqns (2), the inplane stretching stiffness A_{ij} and out-of-plane bending stiffness D_{ij} are defined by

$$A_{ij} = \sum_{k=1}^{K} \bar{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{K} \bar{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3) \tag{3}$$

where z_k is a distance from the middle surface to upper surface of the k-th layer ($k = 1, 2, \ldots, K$) as shown in Fig. 2. The $\bar{Q}_{ij}^{(k)}$ are elastic constants in the k-th layer defined with respect to the x and y axis, and are converted by considering the fiber orientation angle θ_k from the orthotropic elastic constants

$$Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}$$

$$Q_{12} = Q_{11} \nu_{21} = Q_{22} \nu_{12}, \quad Q_{66} = G_{12} \tag{4}$$

with E_1, E_2, G_{12} and ν_{12} being modulus of elasticity defined along the material principal axes, i.e. axes taken in the fiber and its transverse directions (cf Jones, 1975; Vinson and Sierakowski, 1986).

The displacements of the middle surface may be expressed by functions

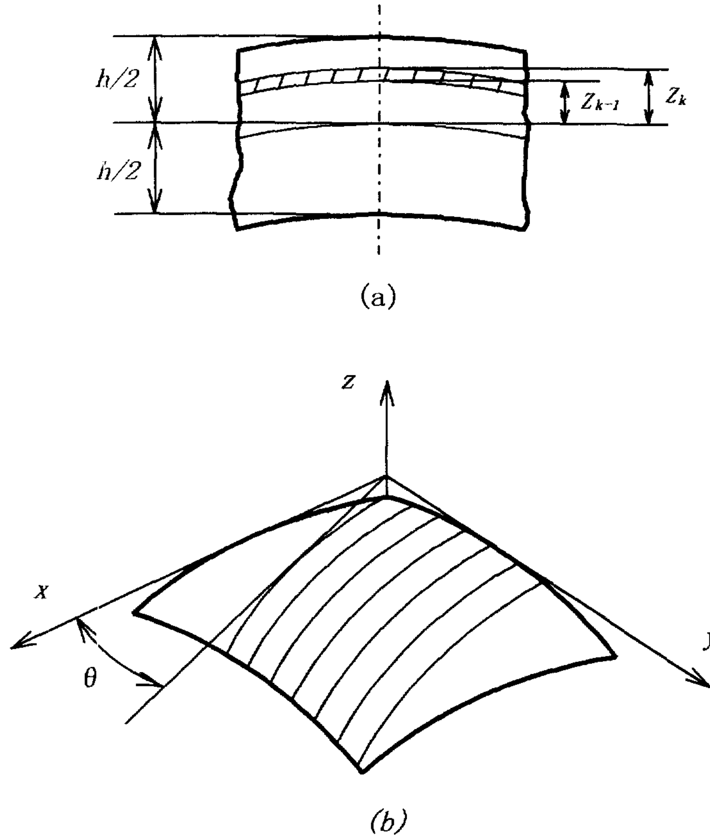

Fig. 2. Laminate configuration of the shell (a) cross-section, (b) fiber orientation angle of the k-th layer.

$$u(x,y,t) = U_{mn} h \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{j\omega t}$$

$$v(x,y,t) = V_{mn} h \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{j\omega t}$$

$$w(x,y,t) = W_{mn} h \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{j\omega t} \tag{5}$$

where m and n indicate half wave numbers of the mode shape along the x and y axes, respectively, and U_{mn}, V_{mn} and W_{mn} are unknown constants representing nondimensional amplitudes. The thickness h is multiplied in eqns (5) to make the constants nondimensional. The equations satisfy the boundary conditions when the four edges are supported by shear diaphragms.

$$v = w = M_x = N_x = 0 \quad \text{along} \quad x = 0, a$$

$$u = w = M_y = N_y = 0 \quad \text{along} \quad y = 0, b \tag{6}$$

where M_x and M_y are bending moments

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \end{Bmatrix} \tag{7}$$

about the y and x axis, respectively, and N_x and N_y are stress resultants

$$\begin{Bmatrix} N_x \\ N_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{w}{R_x} \\ \dfrac{\partial v}{\partial y} + \dfrac{w}{R_y} \end{Bmatrix} \tag{8}$$

in the x and y directions. Substitution of assumed solutions (5) into (1) yields an eigenvalue equation

$$[K]\{u\} = \Omega^2\{u\} \tag{9}$$

where $\{u\} = \{U_{mn}, V_{mn}, W_{mn}\}^{\mathrm{T}}$ and the elements K_{ij} in a symmetric matrix $[K]$ are given by

$$\begin{aligned}
K_{11} &= \bar{A}_{11}\bar{m}^2 + \bar{A}_{66}\bar{n}^2 \\
K_{12} &= K_{21} = (\bar{A}_{12} + \bar{A}_{66})\bar{m}\bar{n} \\
K_{13} &= K_{31} = -(\bar{A}_{11} + \bar{A}_{12}\gamma)\beta\bar{m} \\
K_{22} &= \bar{A}_{66}\bar{m}^2 + \bar{A}_{22}\bar{n}^2 \\
K_{23} &= K_{32} = -(\bar{A}_{12} + \bar{A}_{22}\gamma)\beta\bar{n} \\
K_{33} &= \bar{D}_{11}\bar{m}^4 + 2(\bar{D}_{12} + 2\bar{D}_{66})\bar{m}^2\bar{n}^2 + \bar{D}_{22}\bar{n}^4 + \beta^2(\bar{A}_{11} + 2\bar{A}_{12}\gamma + \bar{A}_{22}\gamma^2)
\end{aligned} \tag{10}$$

Nondimensional quantities in eqns (10) are defined by

$$\begin{aligned}
&\alpha = \frac{a}{b}, \quad \beta = \frac{a}{R_x}, \quad \gamma = \frac{R_x}{R_y} \\
&\bar{m} = m\pi, \quad \bar{n} = \alpha n\pi \\
&\bar{A}_{ij} = \frac{A_{ij}a^2}{D_0}, \quad \bar{D}_{ij} = \frac{D_{ij}}{D_0} \\
&\Omega^2 = \frac{\rho\omega^2 a^4}{D_0} \quad \text{(nondimensional frequency parameter)} \\
&D_0 = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})} \quad \text{(reference stiffness)}
\end{aligned} \tag{11}$$

When $\{u\}_{mn}$ denotes an (m, n)-th normalized eigenvector to the eigenvalue eqn (9), the corresponding frequency parameter may be written as

$$f_{mn} = \Omega_{mn}^2 = \{u\}_{mn}^{\mathrm{T}}[K]\{u\}_{mn} \tag{12}$$

3. DESIGN METHOD FOR OPTIMIZATION

3.1. *Mathematical model used in the optimization*

In developing a design method applied to the free vibration problem of symmetrically laminated shallow shells, an object function is taken to be the square of fundamental frequency $f = \min(\Omega_{mn}^2)$, whose maximum values are searched with respect to the design variables of fiber orientation angles θ_k. The term "fundamental frequency" referred here indicates the lowest eigenvalue for given conditions. A vibration mode is defined by a pair of integers (m, n), where m and n are the half wave numbers in assumed solution (5). It is noted in the present problem that the fundamental vibration mode is not limited to $(m, n) = (1,1)$ and may take on other wave numbers, such as (1,2) or (2,1). This is often found in shell problems by the effect of geometric curvatures.

When the design variables are written in the form

$$x_k = \cos(\pm 2\theta_k) \quad (k = 1, 2, \ldots, K) \tag{13}$$

the present problem is mathematically formulated in the following form

$$\begin{aligned}
&\text{Find } \mathbf{x} = [x_1, x_2, \ldots, x_K]^{\mathrm{T}} \\
&\text{maximize } \{\min(\{u\}_{mn}^{\mathrm{T}}[K]\{u\}_{mn})\} \quad (m, n = 1, 2, \ldots) \\
&\text{subject to } -1 \leqslant x_k \leqslant 1 \quad (k = 1, 2, \ldots, K)
\end{aligned} \tag{14}$$

The second equation in eqns (14) implies that a solution point is sought that maximizes the lowest value of f_{mn} for various m and n.

3.2. *Stationary solution for a single* (m, n) *mode*

Next, we apply the Kuhn–Tucker optimality condition directly to the frequency curves. It is well known that this condition is a stationary condition, and generally speaking solution points that satisfy this condition fall into local maximum, local minimum or saddle point of the object function. Here this condition is applied to the frequency curve with a fixed (m, n), i.e. a single mode. The maximum point for a single mode is found by choosing the largest value among the obtained stationary points.

Suppose that $\mathbf{x}^s$ is a stationary solution for a single mode, it must satisfy the Kuhn–Tucker optimality condition with Lagrange multipliers λ_k.

$$\begin{gathered}
\nabla f(\mathbf{x}^s) + \sum_{k=1}^{2K} \lambda_k \nabla g_k(\mathbf{x}^s) = 0 \\
\sum_{k=1}^{2K} \lambda_k g_k(\mathbf{x}^s) = 0 \\
\lambda_k \geqslant 0, \quad g_k(\mathbf{x}^s) \geqslant 0 (k = 1, 2, \ldots, 2K)
\end{gathered} \tag{15}$$

where

$$g_{2k-1}(\mathbf{x}^s) = 1 - x_k, \quad g_{2k}(\mathbf{x}^s) = 1 + x_k \quad (k = 1, 2, \ldots, K)$$

When the range of $x_k \leqslant -1$ is considered, then x_k^s is exactly $x_k^s = -1$ because of the range of cosine function (13). The same is true for $x_k^s = 1$ of $1 \leqslant x_k$. For $-1 < x_k < 1$, the constraints are non-active and the corresponding Lagrange multipliers λ_k are zero. Hence, solving a set of eqns (15) yields

$$\begin{aligned}
&\frac{\partial f(\mathbf{x}^s)}{\partial x_k} = 0 \quad (-1 < x_k < 1) \\
&x_k^s = -1 \quad (x_k \leqslant -1) \\
&x_k^s = 1 \quad (1 \leqslant x_k) \quad (k = 1, 2, \ldots, 2K)
\end{aligned} \tag{16}$$

After an eigenvalue eqn (9) is differentiated and arranged, formulas are obtained to calculate sensitivity of the object function f

$$\frac{\partial f}{\partial x_k} = \{u\}_{mn}^{\mathrm{T}} \left[\frac{\partial K}{\partial x_k}\right] \{u\}_{mn} \tag{17}$$

Under an assumption that each layer is of the same material, the following invariant quantities are introduced, with the elastic constants Q_{ij} defined in eqns (4), by

$$\begin{aligned} U_1 &= \tfrac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}) \\ U_2 &= \tfrac{1}{8}(4Q_{11}-4Q_{22}) \\ U_3 &= \tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{66}) \\ U_4 &= \tfrac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{66}) \\ U_5 &= \tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}+4Q_{66}) \end{aligned} \tag{18}$$

By using quantities defined in (18), the stretching stiffness A_{ij} and the bending stiffness D_{ij} are rewritten as

$$\begin{aligned} A_{11} &= (U_1-U_3)h+\sum_{k=1}^{K}(2U_3x_k^2+U_2x_k)(z_k-z_{k-1}) \\ A_{22} &= (U_1-U_3)h+\sum_{k=1}^{K}(2U_3x_k^2-U_2x_k)(z_k-z_{k-1}) \\ A_{12} &= (U_3+U_4)h-2U_3\sum_{k=1}^{K}x_k^2(z_k-z_{k-1}) \\ A_{66} &= (U_3+U_5)h-2U_3\sum_{k=1}^{K}x_k^2(z_k-z_{k-1}) \\ D_{11} &= \frac{h^3}{12}(U_1-U_3)+\frac{1}{3}\sum_{k=1}^{K}(2U_3x_k^2+U_2x_k)(z_k^3-z_{k-1}^3) \\ D_{22} &= \frac{h^3}{12}(U_1-U_3)+\frac{1}{3}\sum_{k=1}^{K}(2U_3x_k^2-U_2x_k)(z_k^3-z_{k-1}^3) \\ D_{12} &= \frac{h^3}{12}(U_3+U_4)-\frac{2}{3}U_3\sum_{k=1}^{K}x_k^2(z_k^3-z_{k-1}^3) \\ D_{66} &= \frac{h^3}{12}(U_3+U_5)-\frac{2}{3}U_3\sum_{k=1}^{K}x_k^2(z_k^3-z_{k-1}^3) \end{aligned} \tag{19}$$

Substitution of eqns (10), (11), (17)–(19) into (16) gives, after some manipulations, a set of equations to obtain stationary solutions x_k for given mode (m, n).

$$P_1\,\Delta z_k+P_2\,\Delta z_k^3+x_k[P_3\,\Delta z_k+P_4\,\Delta z_k^3]=0 \quad (k=1,2,\ldots,K) \tag{20}$$

where

$$\begin{aligned} P_1 &= U_2[\bar{m}^2u_1^2-\bar{n}^2u_2^2+\beta^2(1-\gamma^2)u_3^2-2\beta(\bar{m}u_1u_3-\bar{n}\gamma u_2u_3)] \\ P_2 &= \tfrac{1}{3}U_2(\bar{m}^4-\bar{n}^4)u_3^2 \\ P_3 &= 4U_3[(\bar{m}^2-\bar{n}^2)(u_1^2-u_2^2)+\beta^2(1-\gamma)^2u_3^2-4\bar{m}\bar{n}u_1u_2-2\beta(1-\gamma)(\bar{m}u_1u_3-\bar{n}u_2u_3)] \\ P_4 &= \tfrac{4}{3}U_3(\bar{m}^4-6\bar{m}^2\bar{n}^2+\bar{n}^4)u_3^2 \end{aligned} \tag{21}$$

with

$$\Delta z_k = (z_k - z_{k-1})/a$$
$$\Delta z_k^3 = (z_k^3 - z_{k-1}^3)/a^3 \tag{22}$$

The u_1, u_2 and u_3 are components of the normalized eigenvector. Equation (20) is not expressed in the explicit form of x_k, and the solution points should be numerically solved to a given pair of (m, n).

3.3. *Optimization process*

The globally maximum solution cannot be uniquely determined only from eqn (20), and two distinct cases are identified through numerical experiments. Figure 3 demonstrates two such typical cases, wherein the frequency parameters Ω for m, $n \leqslant 2$ are presented against a fiber orientation angle θ for the shell considered in the next section. Figure 3(a) shows a case when a frequency curve of the (1,1) mode does not cross with those of different

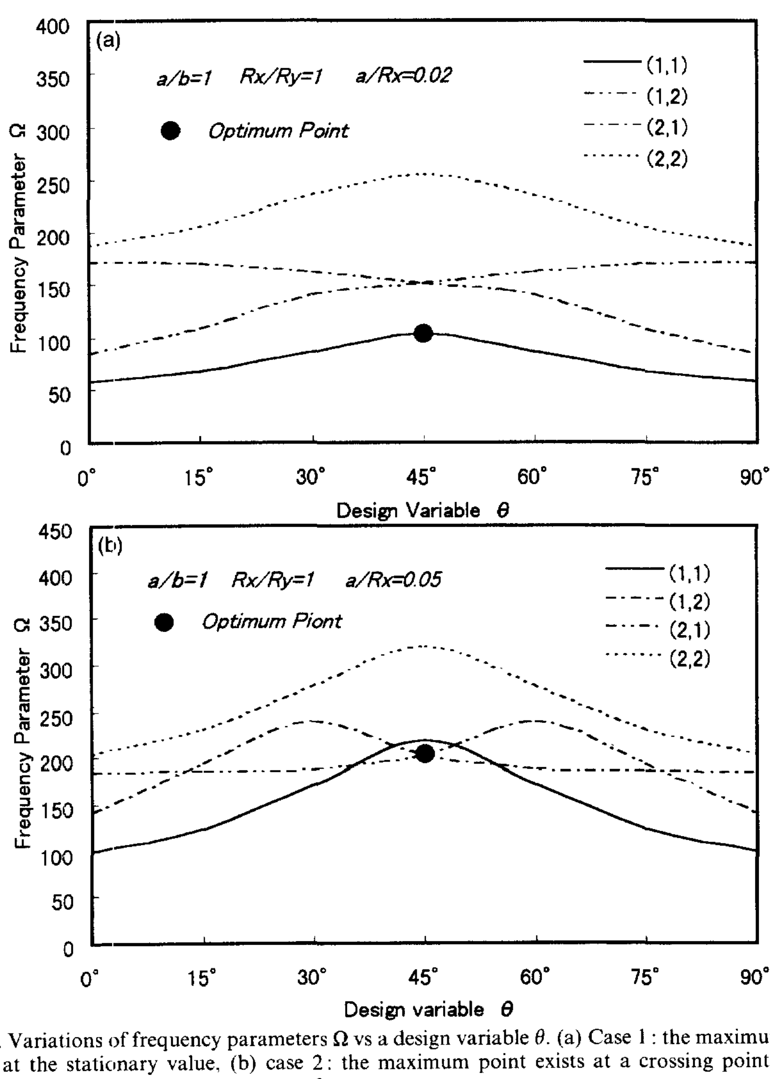

Fig. 3. Variations of frequency parameters Ω vs a design variable θ. (a) Case 1 : the maximum point exists at the stationary value, (b) case 2 : the maximum point exists at a crossing point of two frequency curves.

wave numbers and the maximum point is found on the single curve. In this case, the maximum is obtained just by solving a single eqn (20) and choosing the largest one among the obtained stationary points.

More complicated is the case shown in Fig. 3(b), wherein a crossing point of two frequencies with $(m, n) = (1,2)$ and $(2,1)$ is the maximum. This solution is not obtained from eqn (20) and this specific point is determined by the bisection method. These two cases are summarized as

Case (1) The maximum solution for a single fixed (m, n) exists at a stationary point of a function of Ω. This solution can be obtained from eqn (20).

Case (2) A possible solution for the maximum point exists at a crossing point of two functions of Ω with different pairs of (m, n) and (m', n') under a condition that

$$\frac{d\Omega_{mn}}{d\theta} \cdot \frac{d\Omega_{m'n'}}{d\theta} \leqslant 0 \tag{23}$$

The crossing point is determined by using the bisection method.

In the calculations to follow, solution points of Case (1) and Case (2) are computed for $(m, n) \leqslant (6, 6)$, and among a finite number of these possible solutions from the two cases, the final (i.e., globally maximum) solution is determined by picking up only one largest value of frequency.

4. NUMERICAL EXAMPLES AND DISCUSSIONS

Using the vibration solution and the design method developed for the problem, optimal solutions are computed and presented in this section. As explained, the shallow shell is supported by shear diaphragm along the edges, and is laminated with layers of equal thickness and of the same composite material. Other conditions introduced in numerical examples are as follows.

(1) A carbon-fiber composite material CFRP (T300/5208) having relatively large degree of orthotropy is considered, and the elastic constants are taken as

$$E_1 = 181\text{ GPa}, \quad E_2 = 10.3\text{ GPa}, \quad G_{12} = 7.17\text{ GPa} \quad \text{and} \quad \nu_{12} = 0.3$$

(2) A symmetric stacking configuration $[(\pm\theta)_6]_s$ is assumed for twelve layered shells. The notation $[(\pm\theta)_6]_s$ means that each layer in the upper (or lower) six layers can take arbitrarily on $+\theta$ or $-\theta$. Such ambiguity is allowed, because the cross elasticity terms are neglected ($A_{16} = A_{26} = D_{16} = D_{26} = 0$) in the vibration eqn (1) and therefore identical frequency values may result in different combinations of layers in $[(\pm\theta)_6]_s$. The thickness ratio is kept constant as $a/h = 400$ in Table 1–3, wherein the frequency parameter $\Omega = \omega a^2(\rho/D_0)^{1/2}$ defined in eqn (11) is used. In Figs 4–6, the thickness ratio $c/h = 400$ ($c^2 = ab$) is used in a new frequency parameter $\Omega = \omega c^2(\rho/D_0)^{1/2}$ which excludes effects of difference in area on the frequencies.
(3) The shell has rectangular planform ($a/b = 0.5$–2) including a square shape ($a/b = 1$).
(4) The curvature ratio is $R_x/R_y = 0$ ($R_x \neq 0, R_y = \infty$) for a cylindrical shell, $R_x/R_y = 1$ for a spherical shell, and $R_x/R_y = -1$ for a hyperbolic paraboloidal shell.

Table 1 presents the optimal (maximum) fundamental frequencies Ω^* and the corresponding design variables θ^* (a symbol "*" indicates that those with * are optimal values) for shallow shells of cylindrical curvature. The aspect ratio a/b and the degree of curvature a/R_x are taken as indicated in the table. The half wave numbers $(m, n)^*$ are presented for the vibration mode that yields the frequencies Ω^*. A single $(m, n)^*$ falls into case (1), while two sets of $(m, n)^*$ are given for case (2). For example in the case of $a/b = 2$, results for $a/R_x = 0.2$ and 0.1 are case (2) where the solutions are found at crossing points of modes "(1,1) and (3,1)" and "(1,1) and (2,1)", respectively. The result of $a/R_x = 0.05$ is case (1) where the solution is found at the maximum of (1,1) mode. Such different features probably

Table 1. Optimal frequency parameters Ω^* and fiber angles θ^* for twelve-layered shallow shells with cylindrical curvature ($R_x/R_y = 0, a/h = 400, [(\pm\theta)_6]_s$)

a/b	a/R_x	Ω^*	θ^*	$(m, n)^*$
2	0.2	439.8	49.70	(1,1), (3,1)
	0.1	311.0	39.20	(1,1), (2,1)
	0.05	203.4	30.15	(1,1)
1.75	0.2	385.0	39.56	(2,1), (3,1)
	0.1	275.3	43.83	(1,1), (2,1)
	0.05	187.9	32.27	(1,1)
1.5	0.2	333.2	45.45	(2,1), (3,1)
	0.1	234.0	50.01	(1,1), (2,1)
	0.05	170.1	35.25	(1,1)
1.25	0.2	272.8	52.35	(2,1), (3,1)
	0.1	188.6	51.46	(2,1)
	0.05	148.6	39.39	(1,1)
1	0.2	203.3	59.85	(2,1)
	0.1	163.6	28.17	(1,1), (2,1)
	0.05	121.4	44.99	(1,1)
0.75	0.2	161.6	20.85	(1,1), (2,1)
	0.1	131.2	40.60	(1,1), (2,1)
	0.05	87.90	52.13	(1,1)
0.5	0.2	114.5	42.10	(1,1), (2,1)
	0.1	85.38	55.72	(1,1), (2,1)
	0.05	50.85	59.84	(1,1)

Table 2. Optimal frequency parameters Ω^* and fiber angles θ^* for twelve-layered shallow shells with spherical curvature ($R_x/R_y = 1, a/h = 400, [(\pm\theta)_6]_s$)

a/b	a/R_x	Ω^*	θ^*	$(m, n)^*$
2	0.2	693.8	37.41	(1,2), (3,1)
	0.1	422.2	33.58	(1,1), (2,1)
	0.05	240.1	29.27	(1,1)
1.75	0.2	653.1	38.91	(1,2), (3,1)
	0.1	408.9	37.26	(1,1), (2,1)
	0.05	233.9	31.52	(1,1)
1.5	0.2	618.8	40.83	(1,2), (3,1)
	0.1	379.2	39.37	(1,2), (2,1)
	0.05	228.5	34.65	(1,1)
1.25	0.2	593.5	43.30	(1,2), (3,1)
	0.1	348.9	41.62	(1,2), (2,1)
	0.05	222.8	40.62	(1,1), (2,1)
1	0.2	565.3	45.00	(1,3), (3,1)
	0.1	327.7	45.00	(1,2), (2,1)
	0.05	203.3	45.00	(1,2), (2,1)
0.75	0.2	546.7	46.85	(1,4), (3,1)
	0.1	308.3	48.02	(1,3), (2,1)
	0.05	185.6	48.96	(1,2), (2,1)
0.5	0.2	527.6	48.30	(1,5), (2,1)
	0.1	293.2	50.48	(1,4), (2,1)
	0.05	173.5	52.59	(1,3), (2,1)

stem from interactions between the one-directional cylindrical curvature of the shell and the curvature of deflected shapes of vibration. It is seen in the table that the optimal frequencies Ω^* decrease monotonically as a/R_x becomes smaller (i.e., the shell gets more shallow), and the corresponding fiber angle θ^* varies considerably.

Tables 2 and 3 are in the same format as Table 1, except that they are calculated for spherical and hyperbolic paraboloidal shells, respectively. In Table 2, it is seen that the optimal frequencies decrease rapidly as a/R_x gets smaller, and the optimal angle for square planform ($a/b = 1$) is 45° for all values of a/R_x. Most of the solutions fall into the case (2), except for most shallow one ($a/R_x = 0.05$) of $a/b = 2$, 1.75 and 1.5.

Table 3. Optimal frequency parameters Ω^* and fiber angles θ^* for twelve-layered shallow shells with hyperboroidal curvature ($R_x/R_y = -1, a/h = 400, [(\pm\theta)_6]_s$)

a/b	a/R_x	Ω^*	θ^*	$(m,n)^*$
2	0.2	240.4	44.97	(2,1)
	0.1	239.7	41.27	(1,1), (2,1)
	0.05	175.3	83.95	(1,1)
1.75	0.2	238.1	46.00	(2,1)
	0.1	218.2	39.85	(1,1), (2,1)
	0.05	146.9	34.52	(1,1)
1.5	0.2	284.3	45.58	(1,1), (2,1)
	0.1	184.2	34.99	(1,1)
	0.05	119.6	37.78	(1,1)
1.25	0.2	199.1	39.13	(1,1)
	0.1	118.7	39.91	(1,1)
	0.05	88.04	42.61	(1,1)
1	0.2	59.91	44.95	(1,1)
	0.1	60.09	44.95	(1,1)
	0.05	60.14	44.97	(1,1)
0.75	0.2	293.5	52.74	(1,1)
	0.1	126.1	52.60	(1,1)
	0.05	73.80	51.63	(1,1)
0.5	0.2	59.91	44.95	(1,2)
	0.1	60.09	44.95	(1,2)
	0.05	60.14	44.97	(1,2)

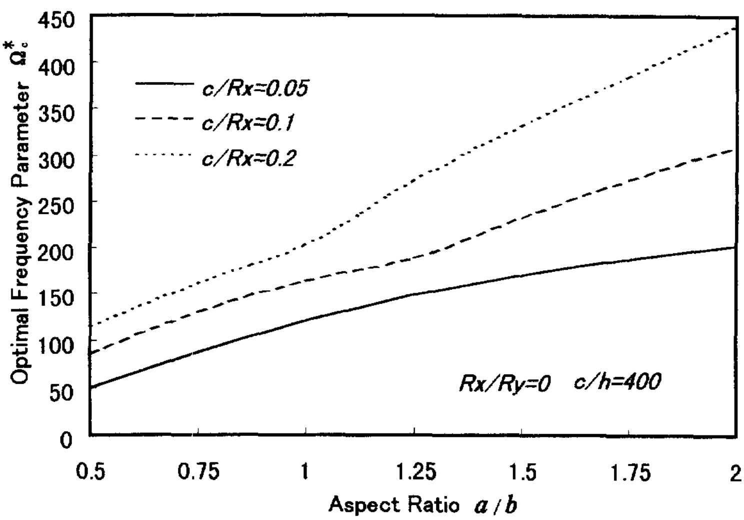

Fig. 4. Variations of optimal solutions Ω_c^* vs an aspect ratio a/b for twelve-layered shallow shells with cylindrical curvature ($R_x/R_y = 0, a/h = 400, [(\pm\theta)_6]_s$).

In contrast, in Table 3 for the hyperbolic paraboloidal shells, all solutions are case (1), except for three of ($a/b = 2, a/R_x = 0.1$), (1.75, 0.1) and (1.5, 0.2). This tendency is quite different from that found in Tables 1 and 2. It may be noted that optimal frequencies obtained for $a/b = 1$ and $a/b = 0.5$ are identical. This is because the straight nodal line appearing on the (1,2) mode of $a/b = 0.5$ plays a role of a simple support along the line and the shell ($a/b = 0.5$) can be regarded as two adjacent shells of square planform. The optimal fiber angles in these cases ($a/b = 1$ and 0.5) should be exactly $\theta^* = 45°$ due to the physical reason, but these values (e.g., $\theta^* = 44.95°$, not exactly $\theta^* = 45°$) were obtained due to a convergence criterion used in the iterative process.

Figures 4, 5 and 6 present the optimal frequencies vs the aspect ratio a/b for shallow shells of cylindrical, spherical and hyperbolic paraboloidal curvatures, respectively. The frequency parameter is redefined here as

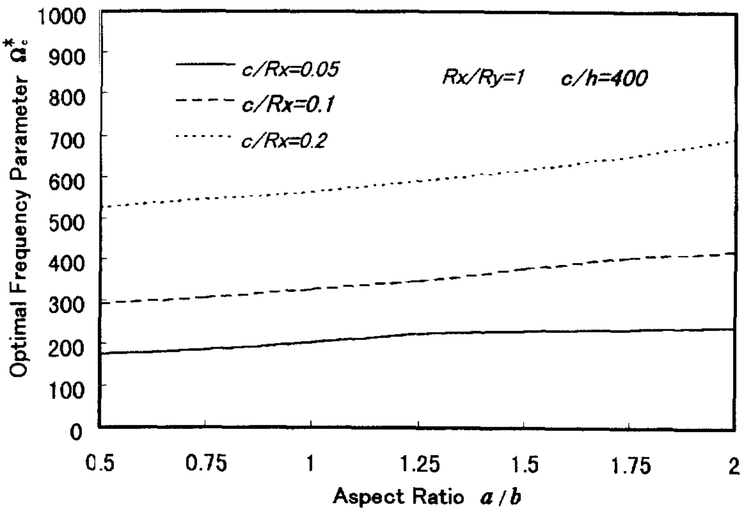

Fig. 5. Variations of optimal solutions Ω_c^* vs an aspect ratio a/b for twelve-layered shallow shells with cylindrical curvature ($R_x/R_y = 1, a/h = 400, [(\pm\theta)_6]_s$).

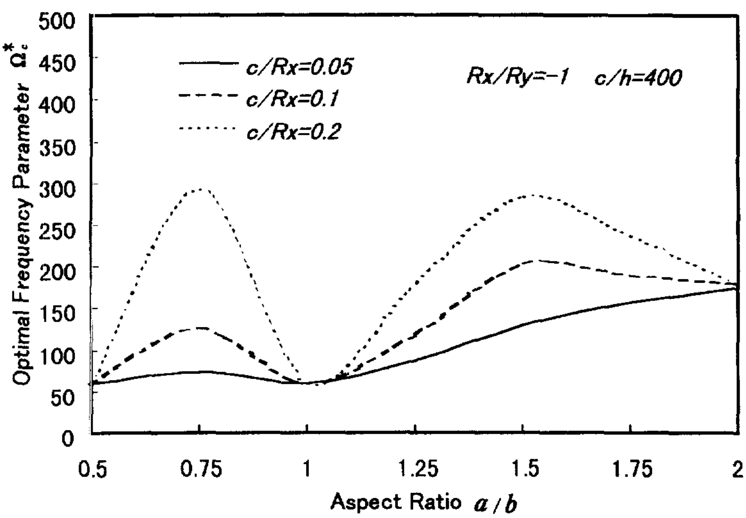

Fig. 6. Variations of optimal solutions Ω_c^* vs an aspect ratio a/b for twelve-layered shallow shells with hyperbolic paraboloidal curvature ($R_x/R_y = -1, a/h = 400, [(\pm\theta)_6]_s$).

$$(\Omega_c)^2 = \frac{\rho\omega^2 c^4}{D_0} \quad \text{(non-dimensional frequency parameter)} \tag{24}$$

by using the equivalent area $c^2 = ab$ even for different aspect ratios a/b. This is done so to exclude effects of difference in area on the frequencies. By the introduction of a reference length c, a new thickness parameter c/h and curvature parameters c/R_x are used in the figures. Solid, broken and dotted lines denote results of $c/R_x = 0.05$, 0.1 and 0.2, respectively.

Optimal solutions of cylindrical shells tend to increase gradually as a/b is increased, as seen in Fig. 4, while those of spherical shells show almost flat lines in Fig. 5. Most interesting is the result for hyperbolic paraboloidal shells in Fig. 6, wherein the optimal frequencies vary in a wave-like manner. This behavior is more pronounced for larger c/R_x. However, those three lines share the same points at $a/b = 0.5$, 1 and 2, since the optimal vibration modes at such a/b can be regarded as an assemblage of the equivalent single mode shape of a shallow shell with square planform, as explained for cases of $a/b = 1$ and 0.5 in Table 3.

5. CONCLUDING REMARKS

An optimal design method is developed for improving the free vibration characteristics of symmetrically laminated shallow shells. Concluding remarks are summarized as below.

(1) An exact frequency equation is derived for shallow shells supported by shear diaphragms along the edges. An optimization problem is formulated in such a way that the fiber orientation angles θ_k are design variables and the fundamental (lowest) frequency parameter is an object function to be maximized. A set of equations are then derived through the sensitivity analysis to determine stationary solutions for a single fixed mode by making a direct use of Kuhn–Tucker optimality condition.

(2) In the shell vibration, the fundamental mode takes various mode shape patterns depending upon the fiber angles and shell curvatures. For such difficulties where lowest mode pattern is not self-evident, two distinct cases of how possible solutions exist are classified through a number of numerical experiments. Two independent approaches for the cases enabled us to determine the global solution in the end.

(3) For symmetrically laminated shallow shells with three different curvatures (cylindrical, spherical and hyperbolic paraboloidal shells), optimal solutions are obtained for different aspect ratios and curvature values. This numerical study revealed that the present approach has a wide applicability to this type of optimization.

(4) Most of optimal solutions for cylindrical and spherical shells are found at crossing points of two frequency functions in the "fiber orientation angle"–"frequency parameter" curves. In contrast, for a hyperbolic paraboloidal shell, most of the solutions exist at the extremum of a single curve with specific mode. It is also found that the solutions for square shells with spherical and hyperbolic paraboloidal curvatures shows the optimal fiber angles of 45°. These effects of shell curvatures were clarified for the first time in the present study, which were not observable in the optimization of flat plate problems.

(5) As expected, the optimal frequencies are higher as the curvature ratio a/R_x gets greater in all shell configurations, except that some identical optimal frequencies are given for hyperbolic paraboloidal shells.

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